

Some Examples of $RS_3^2(3)$ -Transformations of Ranks 5 and 6 as the Higher Order Transformations for the Hypergeometric Function

F. V. Andreev and A. V. Kitaev *

Steklov Mathematical Institute, Fontanka 27, St.Petersburg, 191011, Russia*

Kansas State University, Department of Mathematics,
Manhattan, KS, 66506, USA

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Abstract

A combination of rational mappings and Schlesinger transformations for a matrix form of the hypergeometric equation is used to construct higher order transformations for the Gauss hypergeometric function.

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*E-mail: andreev@math.ksu.edu; akitaev@pdmi.ras.ru

1 Introduction

This is a continuation of our series of works devoted to the so-called RS -transformations for linear matrix ODEs. The notation $RS_p^n(q)$ means RS -transformation of $n \times n$ matrix linear ODE with q singular points into another linear ODE with $p \geq q$ singular points. The abbreviation RS is related with the fact that these transformations are compositions of rational transformations of arguments of the linear ODEs with the Schlesinger transformations of their solutions. The rank of the RS -transformation is by definition the order (a number of preimages) of its rational transformation. It would be very interesting to study these transformations for $n > 2$, however in this, as well as in our previous works, we restrict our consideration to the case of 2×2 matrices and sometimes omit the superscript $n = 2$ to simplify the notation.

In the work [4] it was shown that transformation $RS_4(4)$ generates quadratic transformation for the sixth Painlevé equation, in [5] a general notion of the RS -transformation for the so-called special functions of the isomonodromy type [6] were introduced and application to the construction of the algebraic solutions to the sixth Painlevé equation was discussed, in [7] we classified all $RS_4(3)$ transformations of the ranks ≤ 4 .

In this work we show that some higher order transformations of the Gauss hypergeometric function can be obtained by means of the $RS_3(3)$ -transformations. More precisely, $RS_3^2(3)$ -transformations of the rank r “generate” so-called transformations of order r for the hypergeometric function. Actually, we believe that one can obtain by this method all the “seed” transformations of the higher order for the hypergeometric function, i.e., few transformations from which all other higher order transformations can be derived via already known transformations for the hypergeometric function.

To the best of our knowledge explicit examples of the higher order transformations for the hypergeometric functions, which are not combinations of the quadratic and cubic transformations, are unknown. The main goal of this paper is to present some explicit examples of such transformations of the orders 5 and 6. We hope to release soon a complete classification of the seed transformations of the orders 4 – 6.

It is written in the book [2], with the reference to the works of Goursat dated back to 1881 and 1938, that transformations for the hypergeometric function of the orders different from 2, 3, 4 and 6 exist only in those cases when it is algebraic, i.e., it corresponds to some case in the famous H. A. Schwartz table of 1873. Actually, this statement is not true since for the hypergeometric function corresponding to the Schwartz parameters $(1/2, 1/2, \nu)$, where ν is an arbitrary number, one can apply proper quadratic or cubic transformations arbitrary number of times. This statement is also not true if we exclude higher order transformations related with iterations of the transformations of the orders ≤ 6 . In fact, in the last Section of this paper, we show that there is an eighth order transformation relating hypergeometric functions whose parameters are not included into the Schwartz table. We know also some other examples of this kind.

The paper is organized as follows. In Section 2 we, following the work [3], write the Euler equation for the Gauss hypergeometric function in the matrix form. Actually solutions of this matrix equation can be parameterized in terms of the hypergeometric functions in a variety of ways (recall 24 Kummer’s solutions for the Euler equation). Clearly, for different transformations it is better to choose those representation of solutions in terms of the hypergeometric series in which these transformations look simpler. We are not doing it here and using the same form of the fundamental solution for all transformations under consideration. However our formulas can be easily adopted for

any other representations of the fundamental solutions. The only difference will be in the right-hand factor in the corresponding RS -transformations, which is denoted for all transformations as the matrix V . In Section 3 we explain basic principles for classification of the $RS_3^2(3)$ -transformations and corresponding notation. In Section 4 a brief account of the quadratic and cubic transformations is given. The following two Sections 5 and 6 are devoted to the examples of the RS -transformations of the ranks 5 and 6, respectively. Finally, in Section 7 we discuss an example of the RS -transformation of the rank 8 mentioned above.

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2 Hypergeometric Equation

Consider the following matrix form of the hypergeometric equation,

$$\frac{d\Phi}{d\lambda} = \left(\frac{A}{\lambda} + \frac{B}{\lambda-1} \right) \Phi, \quad (2.1)$$

where we, following [3], parameterize the matrices A and B by three complex numbers, α , β , and δ ,

$$A = \begin{pmatrix} -\frac{(\alpha+\beta)(1-\delta)+2\alpha\beta}{2\beta-2\alpha} & \frac{\beta(\beta+1-\delta)}{\beta-\alpha} \\ -\frac{\alpha(\alpha+1-\delta)}{\beta-\alpha} & \frac{(\alpha+\beta)(1-\delta)+2\alpha\beta}{2\beta-2\alpha} \end{pmatrix}, \quad B = -A + \frac{\beta-\alpha}{2}\sigma_3.$$

Explicit formula for the fundamental solution Φ in terms of the Gauß hypergeometric functions, as well as asymptotic behavior of Φ in the neighborhood of the singular points, can be found in [3]. We'll use these formulas and therefore present them below for reference.

$$\begin{aligned} \Phi(\lambda; \theta_0, \theta_1, \theta_\infty) = & \quad (2.2) \\ & \begin{pmatrix} F(\alpha, \alpha - \delta + 1; \alpha - \beta; \frac{1}{\lambda}) & \frac{\beta(\beta-\delta+1)}{(\beta-\alpha)(\beta-\alpha+1)} \frac{1}{\lambda} \\ \frac{\alpha(\alpha-\delta+1)}{(\alpha-\beta)(\alpha-\beta+1)} \frac{1}{\lambda} & F(\beta, \beta - \delta + 1; \beta - \alpha; \frac{1}{\lambda}) \end{pmatrix} \times \\ & \times F(\alpha + 1, \alpha - \delta + 2; \alpha - \beta + 2; \frac{1}{\lambda}) \quad F(\beta, \beta - \delta + 1; \beta - \alpha; \frac{1}{\lambda}) \\ & \times \lambda^{-\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}} \lambda^{\frac{\delta-1}{2}} (\lambda-1)^{\frac{\alpha+\beta-\delta+1}{2}}, \end{aligned}$$

where $F(\cdot, \cdot; \cdot; \cdot)$ is the Gauss hypergeometric function [2] We define parameters of the formal monodromy, θ_k , as follows:

$$\theta_0 = 1 - \delta, \quad \theta_1 = -\alpha - \beta + \delta - 1, \quad \theta_\infty = \alpha - \beta. \quad (2.3)$$

The inverse formulas read:

$$\alpha = \frac{1}{2}(-\theta_0 - \theta_1 + \theta_\infty), \quad \beta = \frac{1}{2}(-\theta_0 - \theta_1 - \theta_\infty), \quad \delta = 1 - \theta_0.$$

The function $\Phi(\lambda; \theta_0, \theta_1, \theta_\infty)$ has the following asymptotic behavior at singular points,

$$\Phi(\lambda; \theta_0, \theta_1, \theta_\infty) \underset{\lambda \rightarrow 0}{=} G_0(\alpha, \beta, \delta)(I + \mathcal{O}(\lambda))\lambda^{\frac{\theta_0}{2}\sigma_3}C_0(\alpha, \beta, \delta) \quad (2.4)$$

$$\underset{\lambda \rightarrow 1}{=} G_1(\alpha, \beta, \delta)(I + \mathcal{O}(\lambda - 1))(\lambda - 1)^{\frac{\theta_1}{2}\sigma_3}C_1(\alpha, \beta, \delta) \quad (2.5)$$

$$\underset{\lambda \rightarrow \infty}{=} \left(I + \mathcal{O}\left(\frac{1}{\lambda}\right) \right) \left(\frac{1}{\lambda} \right)^{\frac{\theta_\infty}{2}\sigma_3}, \quad (2.6)$$

where $I = \text{diag}\{1, 1\}$ and $\sigma_3 = \text{diag}\{1, -1\}$ are 2×2 diagonal matrices,

$$G_0(\alpha, \beta, \delta) = \frac{1}{\beta - \alpha} \begin{pmatrix} \beta - \delta + 1 & \beta \\ \alpha - \delta + 1 & \alpha \end{pmatrix}, \quad G_1(\alpha, \beta, \delta) = \frac{1}{\beta - \alpha} \begin{pmatrix} 1 & \beta(\beta - \delta + 1) \\ 1 & \alpha(\alpha - \delta + 1) \end{pmatrix},$$

$$C_0(\alpha, \beta, \delta) = e^{\frac{i\pi}{2}(\alpha + \beta - \delta + 1)} \begin{pmatrix} e^{-\pi i(\alpha - \delta + 1)} \frac{\Gamma(\delta - 1)\Gamma(\alpha - \beta + 1)}{\Gamma(\delta - \beta)\Gamma(\alpha)} & -e^{-\pi i(\beta - \delta + 1)} \frac{\Gamma(\delta - 1)\Gamma(\beta - \alpha + 1)}{\Gamma(\delta - \alpha)\Gamma(\beta)} \\ e^{-\pi i\alpha} \frac{\Gamma(1 - \delta)\Gamma(\alpha - \beta + 1)}{\Gamma(1 - \beta)\Gamma(\alpha - \delta + 1)} & -e^{-\pi i\beta} \frac{\Gamma(1 - \delta)\Gamma(\beta - \alpha + 1)}{\Gamma(1 - \alpha)\Gamma(\beta - \delta + 1)} \end{pmatrix},$$

$$C_1(\alpha, \beta, \delta) = \begin{pmatrix} -\frac{\Gamma(\alpha + \beta - \delta + 1)\Gamma(\alpha - \beta + 1)}{\Gamma(\alpha - \delta + 1)\Gamma(\alpha)} & \frac{\Gamma(\alpha + \beta - \delta + 1)\Gamma(\beta - \alpha + 1)}{\Gamma(\beta - \delta + 1)\Gamma(\beta)} \\ -\frac{\Gamma(-\alpha - \beta + \delta - 1)\Gamma(\alpha - \beta + 1)}{\Gamma(1 - \beta)\Gamma(\delta - \beta)} & \frac{\Gamma(-\alpha - \beta + \delta - 1)\Gamma(\beta - \alpha + 1)}{\Gamma(1 - \alpha)\Gamma(\delta - \alpha)} \end{pmatrix}.$$

3 *RS*-transformations

For any matrix linear ODE, e.g., the hypergeometric equation from the previous Section, one can consider, as we call them, *RS*-transformations. These transformations map a given ODE into another ODE and consist of transformations of two types, which we call for brevity *R*- and *S*-transformations. The first one (*R*-) is a rational transformation of the argument of the ODE, and the name *S*-transformation stands for the subsequent Schlesinger transformation of the solution. In some cases it is possible to construct auto-*RS*-transformations which map a given ODE to itself. In this paper we deal with auto-*RS*-transformations for Eq. (2.1). The idea is to find a proper *R*-transformation for a given ODE so that, after a special choice of the parameters of formal monodromy, the parameters θ_0 , θ_1 , and θ_∞ of Section 2, one gets in the transformed ODE $3 + N$ singular points, where N of them are apparent. Therefore, they can be removed via suitable *S*-transformations.

It is clear that as far as a proper *R*-transformation is found one can always construct *S*-transformation which brings *R*-transformed Eq. (2.1) into its original form. Therefore, it is very important to classify all proper *R*-transformations. For this purpose, it is useful to introduce a notion of the rank, r , of the *RS*-transformation, which is equal to the order (a number of preimages) of the corresponding rational function. The rational transformations can be enumerated in terms of the triples of partitions of r which correspond to the multiplicities of the preimages of the points 0, 1, and ∞ . For example, we denote as $R(3 + 2|2 + 2 + 1|4 + 1)$ a rational transformation, $\lambda = R(\lambda_1)$, such that $\lambda = 0$ has two preimages of multiplicities 3 and 2, $\lambda = 1$ has three preimages of multiplicities 2, 2, and 1, and $\lambda = \infty$ has two preimages of multiplicities 4 and 1. Obviously, not for any triple of the partitions one can construct corresponding *R*-transformation, at the same time many triples for which *R*-transformation exists are useless for the construction of the *RS*-transformations, since they do not generate a proper number of the apparent singularities. For low orders of $r \leq 6$ it is easy (relatively) to construct all possible

R -transformations explicitly, whilst for $r \geq 7$ one can make in many cases only theoretical conclusion concerning existence of such transformation, whilst its explicit form could remain unknown. As far as a proper R -transformation is constructed one can make, sometimes, few different choices of the parameters of formal monodromy to generate a desired number of the apparent singularities and therefore one arrives to different RS -transformations related with the given R -transformation. To reflect this we use the following notation for RS -transformations for the Fuchsian ODEs with three singular points,

$$RS_k^n \left(\begin{array}{c|c|c} \theta_0 & \theta_1 & \theta_\infty \\ p_0(r) & p_1(r) & p_\infty(r) \end{array} \right),$$

where n stands for matrix dimension of the ODE under consideration, k denotes a number of non-apparent singularities in the R -transformed ODE, and $p_l(r)$, $l = 0, 1, \infty$, are the corresponding partitions of r . When we refer to all RS -transformations which are associated with the same R -transformation, we denote them as $RS_k^n(p_0(r)|p_1(r)|p_\infty(r))$, finally, we denote just $RS_k^n(m)$ the whole set of RS -transformations of the Fuchsian ODE with m singular points to the one with k singular points.

It is clear that each RS -transformation generates some higher order transformations for the hypergeometric function. On the other hand, many quadratic and cubic transformations were obtained by a variety of different ad hoc methods, in particular, related with the Euler hypergeometric equation. The form of these transformations, all of them are linear with respect to the hypergeometric function, makes natural to suppose that all of them can be derived from RS -transformations of the rank $r \leq 3$. It should be noted that each RS -transformation is only a starting point for construction of a number of the higher order transformations for the hypergeometric function due to the fact that for this function there is a list of the certain transformations and it is not a singlevalued function in \mathbb{C} . For example, consider RS -transformations of Eq. (2.1) of the rank 1. They actually coincide with the corresponding R -transformations of order 1, since they are just fractional-linear transformations of Eq. (2.1) generated by the following two elementary mappings: $\lambda \rightarrow 1/\lambda$ and $\lambda \rightarrow 1 - \lambda$, which produce no any additional singularities in Eq. (2.1) and therefore there is no need to apply any S -transformations. These R -/ RS -transformations of the rank 1 clearly “explain” the relations between the hypergeometric series representing Kummer solutions, however additional efforts should be cast to get these relations explicitly. Even if one knows explicit form of the two “generating” relations, to get the whole list, which is very important for analytical continuation of the solutions, is not absolutely straightforward. Looking at the quadratic transformations (see [2]) one can easily notice, that actually there is only one seed transformation, whilst all the others can be obtained via the linear transformations, relations between the Kummer solutions, the Gauss relations between adjacent hypergeometric functions, and taking the inverse of the R -transformations. The number of cubic transformations is also can be reduced to few seed transformations. Although, we haven’t checked yet the whole Goursat list it is very likely that the seed transformations are exactly those which are generated by the corresponding RS -transformations of the rank 3.

Having in mind the discussion made in the previous paragraph, it seems reasonable to suggest the following scheme for classification of $RS_3^2(3)$ -transformations which generate the seed higher order transformations for the hypergeometric function.

We call two R -transformations $\lambda = R(\lambda_1)$ and $\hat{\lambda} = \hat{R}(\hat{\lambda}_1)$ equivalent if there exist two fractional-linear transformations f and f_1 such that $\hat{\lambda} = f(\lambda)$ and $\hat{\lambda}_1 = f_1(\lambda_1)$.

It is convenient to think about RS -transformations in terms of the parameters of formal monodromy, i.e., in terms of the triples $(\theta_0, \theta_1, \theta_\infty)$. These parameters, completely define Eq. (2.1) and, via Eq. (2.2), corresponding hypergeometric functions. Since with each R -transformation one can associate an infinite number of RS -transformations, via application of the auto- S -transformations to Eq. (2.1), it is reasonable to classify RS -transformations modulo S -transformations: the corresponding hypergeometric functions are related via the Gauss relations for the adjacent functions or their iterations. The action of S -transformations on the triples is as follows

$$\hat{\theta}_l = \theta_l + n_l, \quad l = 0, 1, \infty, \quad n_l \in \mathbb{Z}, \quad n_0 + n_1 + n_\infty = 0 \pmod{2}. \quad (3.1)$$

Another simple transformation whose action is well known is the fractional-linear transformations of λ . The action of the generating transformations $\lambda \rightarrow 1/\lambda$ and $\lambda \rightarrow 1 - \lambda$ on the triples are

$$(\theta_0, \theta_1, \theta_\infty) \rightarrow (\theta_\infty, \theta_1, \theta_0), \quad \text{and} \quad (\theta_0, \theta_1, \theta_\infty) \rightarrow (\theta_1, \theta_0, \theta_\infty). \quad (3.2)$$

Consider the following transformation of Eq. (2.1),

$$\Phi_1(\lambda) = \sigma_1 \Phi(\lambda) \sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This transformation can be interpreted in terms of the formal monodromies twofold:

1) as the transformation,

$$(\theta_0, \theta_1, \theta_\infty) \rightarrow (\theta_0, \theta_1, -\theta_\infty), \quad (3.3)$$

which is equivalent to $(\alpha, \beta, \delta) \rightarrow (\beta, \alpha, \delta)$; and 2)

$$(\theta_0, \theta_1, \theta_\infty) \rightarrow (-\theta_0, \theta_1, -\theta_\infty) \quad (3.4)$$

or $(\alpha, \beta, \delta - 1) \rightarrow (\beta - \delta + 1, \alpha - \delta + 1, 1 - \delta)$. Therefore, one can always fix without lost of generality: $\theta_0 > 0$ and $\theta_\infty > 0$. There is one more analogous transformation of Eq. (2.1),

$$\Phi_1(\lambda) = \begin{pmatrix} 0 & \beta_0 \\ \alpha_0 & 0 \end{pmatrix} \Phi(\lambda) \begin{pmatrix} 0 & 1/\alpha_0 \\ 1/\beta_0 & 0 \end{pmatrix}, \quad \alpha_0 = \alpha(\alpha - \delta + 1), \quad \beta_0 = \beta(\beta - \delta + 1).$$

The latter transformation results in the following transformation of the formal monodromy,

$$(\theta_0, \theta_1, \theta_\infty) \rightarrow (\theta_0, -\theta_1, -\theta_\infty), \quad (3.5)$$

or $(\alpha, \beta, \delta) \rightarrow (-\alpha + \delta - 1, -\beta + \delta - 1, \delta)$. Therefore, we can restrict our consideration only to the case of positive formal monodromies, $\theta_0 > 0$, $\theta_1 > 0$, and $\theta_\infty > 0$.

We call two RS -transformations of Eq. (2.1) equivalent if they are related with the equivalent R -transformations and their corresponding (modulo Eq. (3.2)) formal monodromies are related via the commutative group of transformations given by Eqs. (3.1), (3.3)–(3.5) and the identical transformation.

Remark 3.1 (Notational comment) *To simplify notation in Subsections of the following Sections we use the same notation for similar quantities corresponding to different transformations. This should not confuse the reader, since the values of these quantities are “locally defined” and valid only inside of the corresponding subsection. At the same time we have global notation valid through the whole paper, this is the notation introduced for the function $\Phi(\lambda; \theta_0, \theta_1, \theta_\infty)$, in Section 2 in Eq. (2.2) and below to the end of the section.*

4 Quadratic and Cubic Transformations

Now we are ready to discuss briefly the seed quadratic and cubic transformations for the hypergeometric function or, by other words, the equivalence classes of RS -transformations of Eq. (2.1) of the ranks 2 and 3.

4.1 Quadratic transformations

It is easy to prove that there is only one class of R -transformations which generates RS -transformations of the rank 2. That is, $R(2|1+1|2)$. This transformation is just

$$\lambda = -4\lambda_1(\lambda_1 - 1) \quad \lambda - 1 = -(2\lambda_1 - 1)^2. \quad (4.1)$$

To this R -transformation corresponds only one class of the RS -transformations, that is $RS_3^2(1+1|2|2)$. In terms of the formal monodromies it reads as follows,

$$(\theta_0, \frac{1}{2}, \theta_\infty) \rightarrow (\theta_0, \theta_0, 2\theta_\infty - 1). \quad (4.2)$$

The complete description of $RS_3(1+1|2|2)$ is as follows:

$$\alpha = \frac{\theta_\infty}{2} - \frac{1}{4} - \frac{\theta_0}{2}, \quad \beta = -\frac{\theta_\infty}{2} - \frac{1}{4} - \frac{\theta_0}{2}, \quad \delta = 1 - \theta_0. \quad (4.3)$$

Corresponding matrices A and B in Eq. (2.1) read,

$$A = \frac{1}{16\theta_\infty} \begin{pmatrix} 1 - 4\theta_0^2 - 4\theta_\infty^2 & 4\theta_0^2 - (1 + 2\theta_\infty)^2 \\ (2\theta_\infty - 1)^2 - 4\theta_0^2 & 4\theta_0^2 + 4\theta_\infty^2 - 1 \end{pmatrix}, \quad B = -A - \frac{1}{2}\theta_\infty\sigma_3. \quad (4.4)$$

The RS -transformation can be written as follows,

$$\Phi \left(\lambda; \theta_0, \frac{1}{2}, \theta_\infty \right) = S(\lambda_1) p^{\frac{\sigma_3}{2}} \Phi_1(\lambda_1; \theta_0, \theta_0, 2\theta_\infty - 1) V, \quad (4.5)$$

where λ is given by Eq. (4.1); λ_1 belongs to a neighborhood of 0; $S(\lambda_1)$ is the matrix defining Schlesinger transformations removing an apparent singularity at $\lambda_1 = 1/2$,

$$S(\lambda_1) = \sqrt{\lambda_1 - \frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{\lambda_1 - \frac{1}{2}}} \begin{pmatrix} \frac{(2\theta_\infty+1)^2-4\theta_0^2}{(2\theta_\infty-1)^2-4\theta_0^2} & 0 \\ 1 & 0 \end{pmatrix};$$

$p = \frac{1}{8\theta_\infty}(2\theta_0 + 1 - 2\theta_\infty)$; the function $\Phi_1(\lambda_1; \theta_0, \theta_0, 2\theta_\infty - 1)$ solves Eq. (2.1) with $\lambda \rightarrow \lambda_1$ and the matrices A and B changed respectively by A_1 and B_1 ,

$$A_1 = \frac{1}{4} \begin{pmatrix} 1 - 2\theta_\infty & 1 - 2\theta_\infty - 2\theta_0 \\ -1 + 2\theta_\infty - 2\theta_0 & 2\theta_\infty - 1 \end{pmatrix}, \quad B_1 = -A_1 - \frac{2\theta_\infty - 1}{2}\sigma_3, \quad (4.6)$$

and is given by Eq. (2.2) with $(\lambda, \alpha, \beta, \delta)$ changed to $(\lambda_1, \alpha_1, \beta_1, \delta_1)$, where

$$\alpha_1 = \theta_\infty - \frac{1}{2} - \theta_0, \quad \beta_1 = -\theta_\infty + \frac{1}{2} - \theta_0, \quad \delta_1 = 1 - \theta_0; \quad (4.7)$$

finally, the matrix V , which is independent of λ_1 , can be calculated by means of asymptotic expansions (2.4)–(2.6). Actually, this matrix can be calculated by comparison of the

asymptotic expansions of the left- and right-hand sides of Eq. (4.5) at any of the points $\lambda_1 = 0, 1$, and ∞ . For example, from the comparison of asymptotics at $\lambda_1 = \infty$ one immediately concludes that V is a diagonal matrix. However, to calculate it explicitly one needs few further terms in asymptotics (2.6) which are not given here. Instead of calculation of the absent terms one can compare corresponding asymptotics at $\lambda_1 = 0$. By doing so we find for the matrix V the following representation,

$$V = C_0^{-1}(\alpha_1, \beta_1, \delta_1) D_0^{-1} 2^{\theta_0 \sigma_3} C_0(\alpha, \beta, \delta) = \begin{pmatrix} v & 0 \\ 0 & \det V/v \end{pmatrix},$$

where (also diagonal!) matrix

$$D_0 = G_0^{-1}(\alpha, \beta, \delta) S(0) p^{\frac{\sigma_3}{2}} G_0(\alpha_1, \beta_1, \delta_1) = \sqrt{\frac{\theta_\infty}{2\theta_\infty - 1 - 2\theta_0}} \begin{pmatrix} \frac{2\theta_\infty + 1 + 2\theta_0}{2\theta_\infty - 1 + 2\theta_0} & 0 \\ 0 & \frac{2\theta_\infty + 1 - 2\theta_0}{2\theta_\infty - 1} \end{pmatrix}$$

and

$$v = \det V 2^{\frac{3}{2} - \theta_\infty} e^{\frac{i\pi}{4}(2\theta_\infty - 1 + 2\theta_0)} \sqrt{\frac{\theta_\infty}{2\theta_\infty - 1 - 2\theta_0}}, \quad \det V = \frac{(2\theta_\infty - 1)^2 - 4\theta_0^2}{(2\theta_\infty + 1)^2 - 4\theta_0^2} e^{-\pi i \theta_0}.$$

In the above formulas the non-diagonal matrices $C_0(\cdot, \cdot, \cdot)$ and $G_0(\cdot, \cdot, \cdot)$ are defined at the end of Section 2. They depend on the parameters given by Eqs. (4.7) and (4.3).

This transformation is the seed one for the whole bunch of the quadratic transformations for the hypergeometric function.

4.2 Cubic transformations

A simple examination of possible R -transformations which generate RS -transformations of the rank 3 shows that there exist only three such transformations that is,

$$1. R(2+1|2+1|2+1): \quad \lambda = \rho \lambda_1 \frac{(\lambda_1 - a)^2}{\lambda_1 - b)^2}, \quad \lambda - 1 = \rho(\lambda_1 - 1) \frac{(\lambda_1 - c)^2}{\lambda_1 - b)^2},$$

$$a = \frac{(3\sqrt{\rho} - 1)(1 + \sqrt{\rho})}{4\rho}, \quad b = \frac{(1 + \sqrt{\rho})^2}{4\sqrt{\rho}}, \quad c = \frac{(1 + \sqrt{\rho})^2}{4\rho}, \quad \rho \in \mathbb{C};$$

$$2. R(3|3|1+1+1): \quad \lambda = -\frac{i}{3\sqrt{3}} \frac{(\lambda_1 - 1/2 - i\sqrt{3}/2)^3}{\lambda_1(\lambda_1 - 1)}, \quad \lambda - 1 = -\frac{i}{3\sqrt{3}} \frac{(\lambda_1 - 1/2 + i\sqrt{3}/2)^3}{\lambda_1(\lambda_1 - 1)};$$

$$3. R(2+1|2+1|3): \quad \lambda = 16\lambda_1 \left(\lambda_1 - \frac{3}{4}\right)^2, \quad \lambda - 1 = 16(\lambda_1 - 1) \left(\lambda_1 - \frac{1}{4}\right)^2.$$

These R -transformations generates the following RS ones:

1. $RS_3^2(2+1|2+1|2+1): \quad (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \rightarrow (\frac{1}{2}, \frac{1}{2}, \frac{1}{2});$
2. $RS_3^2(3|3|1+1+1): \quad (\frac{1}{3}, \frac{1}{3}, \theta_\infty) \rightarrow (\theta_\infty, \theta_\infty, \theta_\infty);$
3. (a) $RS_3^2 \left(\begin{array}{c|c|c} 1/2 & \theta_1 & 1/3 \\ 2+1 & 2+1 & 3 \end{array} \right): \quad (\frac{1}{2}, \theta_1, \frac{1}{3}) \rightarrow (\frac{1}{2}, \theta_1, 2\theta_1);$

$$(b) \ RS_3^2 \left(\begin{array}{c|c|c} 1/2 & 1/2 & \theta_\infty \\ 2+1 & 2+1 & 3 \end{array} \right) : \quad (\tfrac{1}{2}, \tfrac{1}{2}, \theta_\infty) \rightarrow (\tfrac{1}{2}, \tfrac{1}{2}, 3\theta_\infty).$$

The cases 1 and 3b are not interesting since they are related with the auto-transformations for the hypergeometric function $F(\frac{\theta_\infty}{2} - \frac{1}{2}, \frac{\theta_\infty}{2}; \theta_\infty; \lambda)$ and its adjacent ones which are known [2] as rather simple functions,

$$F(\frac{\theta_\infty}{2} - \frac{1}{2}, \frac{\theta_\infty}{2}; \theta_\infty; \lambda) = \left(\frac{1}{2} + \frac{1}{2} \sqrt{1-\lambda} \right)^{1-\theta_\infty}.$$

The RS -transformations described in the items 1-3 seems to be the seed ones for the whole bunch of the cubic transformations for the hypergeometric function. We omit exact formulas for the action of these RS -transformations on the function $\Phi(\lambda)$, however, it is interesting to notice that this action is defined via the formulas quite similar to the one for the quadratic case (4.5), with only one, so-called elementary, S -transformation, i.e., one can say that the cubic transformations have the same type of complexity as the quadratic. This, possibly, explains why quadratic and cubic transformations were known for already more than 120 years, whilst the higher order ones the reader will find only in the following Sections.

5 $RS_3^2(5|2+2+1|3+1+1)$

As it is explained in the previous sections the first step in construction of any RS -transformation for Eq. (2.1) is to build an appropriate rational transformation of the variable λ .

5.1 $R(5|2+2+1|3+1+1)$

The rational transformation of λ , which we denote as $R(5|2+2+1|3+1+1)$ reads,

$$\lambda = \frac{(\lambda_1 - a)^5}{\lambda_1(\lambda_1 - 1)(\lambda_1 - b)^3}, \quad \lambda - 1 = \kappa \frac{(\lambda_1 - d)^2(\lambda_1 - c)^2}{\lambda_1(\lambda_1 - 1)(\lambda_1 - b)^3}, \quad (5.1)$$

where

$$\begin{aligned} a &= -\frac{961}{135} + \frac{176}{27}\zeta + \frac{121}{50}\zeta^2 + \frac{11}{50}\zeta^3, & b &= \frac{1817}{81} - \frac{1520}{81}\zeta - \frac{209}{30}\zeta^2 - \frac{19}{30}\zeta^3, \\ d &= \frac{189}{85} - \frac{848}{425}\zeta - \frac{567}{850}\zeta^2 - \frac{249}{4250}\zeta^3, & c &= \frac{66919}{20655} - \frac{188336}{103275}\zeta - \frac{5743}{7650}\zeta^2 - \frac{2689}{38250}\zeta^3, \\ \kappa &= \frac{935}{9} - \frac{800}{9}\zeta - 33\zeta^2 - 3\zeta^3, \end{aligned}$$

and ζ is an arbitrary root of the following equation,

$$27\zeta^4 + 270\zeta^3 + 530\zeta^2 - 1600\zeta + 800 = 0. \quad (5.2)$$

Solution of Eq. (5.2) are as follows,

$$\zeta = -\frac{5}{2} + \varepsilon_1 \frac{35}{18} \sqrt{3} + i \left(\varepsilon_2 \frac{5}{18} \sqrt{15} + \varepsilon_3 \frac{1}{2} \sqrt{5} \right), \quad \varepsilon_k = \pm 1, \quad k = 1, 2, 3, \quad \varepsilon_1 \varepsilon_2 \varepsilon_3 = -1.$$

For example, in the case $\varepsilon_1 = \varepsilon_2 = +1$ and $\varepsilon_3 = -1$ one finds,

$$\begin{aligned} a &= \frac{1}{2} - i\frac{11}{90}\sqrt{15}, & b &= \frac{1}{2} + i\frac{19}{54}\sqrt{15}, & \kappa &= i\frac{5}{3}\sqrt{15}, \\ d &= \frac{1}{2} - \frac{128}{405}\sqrt{3} + i\frac{29}{810}\sqrt{15}, & c &= \frac{1}{2} + \frac{128}{405}\sqrt{3} + i\frac{29}{810}\sqrt{15}. \end{aligned}$$

According to what is written in Section 3 we can associate with this R -transformation two different classes of the RS -transformations related with the following choice of the θ -triples:

$$\left(\frac{1}{5}, \frac{1}{2}, \frac{1}{3}\right), \quad \text{and} \quad \left(\frac{2}{5}, \frac{1}{2}, \frac{1}{3}\right).$$

We consider them in detail in the following subsections.

5.2 $RS_3^2 \left(\begin{array}{c|c|c} 1/5 & -1/2 & 1/3 \\ \hline 5 & 2+2+1 & 3+1+1 \end{array} \right)$

The choice of the parameters corresponding to this transformation is as follows,

$$\alpha = \frac{19}{60}, \quad \beta = -\frac{1}{60}, \quad \delta = \frac{4}{5}.$$

Corresponding matrices A and B in Eq. (2.1) read,

$$A = \frac{1}{1200} \begin{pmatrix} 89 & 11 \\ 589 & -89 \end{pmatrix}, \quad B = \frac{1}{1200} \begin{pmatrix} -289 & -11 \\ -589 & 289 \end{pmatrix}. \quad (5.3)$$

The RS -transformation can be written as follows,

$$\Phi \left(\lambda; \frac{1}{5}, -\frac{1}{2}, \frac{1}{3} \right) = S_1(\lambda_1) S_2(\lambda_1) Q_{BP}^{\frac{\sigma_3}{2}} \Phi_1 \left(\lambda_1; \frac{1}{3}, \frac{1}{3}, \frac{1}{2} \right) V, \quad (5.4)$$

where λ is given by the rational transformation (5.1); λ_1 belongs to a neighborhood of 0, which does not contain the points a, b, c, d , and 1; $S_1(\lambda_1)$ and $S_2(\lambda_1)$ are the matrices defining Schlesinger transformations removing apparent singularities at the points d, a and b, c respectively,

$$\begin{aligned} S_1(\lambda_1) &= \mu_1 J_{da} + 1/\mu_1 J_{ad}, & \mu_1 &= \sqrt{\frac{\lambda_1 - d}{\lambda_1 - a}}, \\ S_2(\lambda_1) &= \mu_2 J_{bc} + 1/\mu_2 J_{cb}, & \mu_2 &= \sqrt{\frac{\lambda_1 - b}{\lambda_1 - c}}, \end{aligned}$$

$$\begin{aligned} J_{ad} &= \frac{1}{20} \begin{pmatrix} 19 & 1 \\ 19 & 1 \end{pmatrix}, & J_{da} &= \frac{1}{20} \begin{pmatrix} 1 & -1 \\ -19 & 19 \end{pmatrix}, \\ J_{bc} &= \frac{1}{20(a-b)} \begin{pmatrix} 19a+d-20b & a-d \\ 19a+d-20b & a-d \end{pmatrix}, & J_{cb} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - J_{bc}, \\ J_{bc}[1,2] &= 1 - J_{bc}[1,1] = \frac{87}{340} - \frac{693}{3400}\zeta - \frac{261}{3400}\zeta^2 - \frac{243}{34000}\zeta^3, \end{aligned}$$

for the choice of ζ given at the end of Subsection 5.1 $J_{bc}[1, 2] = 1/60 + i\sqrt{5}/150$; the matrix

$$Q_B = \frac{\sqrt{66}}{60} \begin{pmatrix} 1 & 1 \\ -\frac{589}{11} & 1 \end{pmatrix},$$

it transforms matrix B to the diagonal form, $Q_B^{-1} B Q_B = \frac{1}{4} \sigma_3$; the number $p = \frac{2057}{945} - \frac{352}{189} \zeta - \frac{121}{175} \zeta^2 - \frac{11}{175} \zeta^3$, in particular, for the choice of the root ζ considered at the end of Subsection 5.1 $p = \frac{i11}{21\sqrt{15}}$; the function $\Phi_1(\lambda_1; \frac{1}{3}, \frac{1}{3}, \frac{1}{2})$ solves Eq. (2.1) with $\lambda \rightarrow \lambda_1$ and the matrices A and B changed respectively by A_1 and B_1 ,

$$A_1 = \frac{1}{24} \begin{pmatrix} -3 & -7 \\ -1 & 3 \end{pmatrix}, \quad B_1 = \frac{1}{24} \begin{pmatrix} -3 & 7 \\ 1 & 3 \end{pmatrix}, \quad (5.5)$$

and is given by Eq. (2.2) with $(\lambda, \alpha, \beta, \delta)$ changed to $(\lambda_1, \alpha_1, \beta_1, \delta_1)$, where

$$\alpha_1 = -\frac{1}{12}, \quad \beta_1 = -\frac{7}{12}, \quad \delta_1 = \frac{2}{3}; \quad (5.6)$$

finally, the matrix V , which is independent of λ_1 , can be calculated by means of asymptotic expansions (2.4)–(2.6). For example, taking the limit $\lambda_1 \rightarrow 0$ in Eq. (5.4) and using asymptotics (2.4) and (2.6) one finds,

$$V = C_0^{-1} (-1/12, -7/12, 2/3) D_0^{-1} q^{\frac{\sigma_3}{6}},$$

where a diagonal (!) matrix

$$D_0 = S_1(0) S_2(0) Q_{BP}^{\frac{\sigma_3}{2}} G_0 (-1/12, -7/12, 2/3) = \begin{pmatrix} 2/\sqrt{-14} & 0 \\ 0 & \sqrt{-14}/3 \end{pmatrix}$$

and

$$q = -\frac{b^3}{a^5} = -300 + 250\zeta + \frac{1485}{16}\zeta^2 + \frac{135}{16}\zeta^3. \quad (5.7)$$

For the choice of the root ζ considered at the end of Subsection 5.1, $q = -\frac{125}{16} - i\frac{75}{16}\sqrt{15}$.

$$C_0^{-1} (-1/12, -7/12, 2/3) = \begin{pmatrix} \frac{\sqrt{\pi}(2-\sqrt{3}+i)\Gamma(\frac{11}{12})}{\sqrt{3}\Gamma(\frac{2}{3})\Gamma(\frac{3}{4})} & \frac{7(2+\sqrt{3}+i)\Gamma(\frac{7}{12})\Gamma(\frac{2}{3})}{24\sqrt{\pi}\Gamma(\frac{3}{4})} \\ -\frac{2(\sqrt{3}+1-i(\sqrt{3}-1))\Gamma(\frac{3}{4})}{7\sqrt{3}\pi\Gamma(\frac{2}{3})\Gamma(\frac{7}{12})} & \frac{(\sqrt{3}-1-i(\sqrt{3}+1))\Gamma(\frac{2}{3})\Gamma(\frac{3}{4})}{12\sqrt{\pi}\Gamma(\frac{11}{12})} \end{pmatrix} \quad (5.8)$$

RS -transformation found in this subsection connects hypergeometric functions corresponding to the second and sixth cases of the Schwartz table.

$$\mathbf{5.3} \quad RS_3^2 \left(\begin{array}{c|c|c} -2/5 & -1/2 & 1/3 \\ 5 & 2+2+1 & 3+1+1 \end{array} \right)$$

The choice of the parameters in Eq. (2.1) is as follows,

$$\alpha = \frac{37}{60}, \quad \beta = \frac{17}{60}, \quad \delta = \frac{7}{5}.$$

Corresponding matrices A and B read,

$$A = \frac{1}{1200} \begin{pmatrix} -19 & 119 \\ 481 & 19 \end{pmatrix}, \quad B = \frac{1}{1200} \begin{pmatrix} -181 & -119 \\ -481 & 181 \end{pmatrix}.$$

The RS -transformation can be written as follows,

$$\Phi\left(\lambda; -\frac{2}{5}, -\frac{1}{2}, \frac{1}{3}\right) = S_1(\lambda_1)S_2(\lambda_1)S_3(\lambda_1)Q_\infty p^{\frac{\sigma_3}{2}} \Phi_1\left(\lambda_1; \frac{1}{3}, \frac{1}{3}, \frac{1}{2}\right) V, \quad (5.9)$$

where λ is given by the rational transformation (5.1); λ_1 belongs to a neighborhood of 0, which does not contain the points a, b, c, d , and 1; $S_1(\lambda_1)$, $S_2(\lambda_1)$, and $S_3(\lambda_1)$ are the matrices defining Schlesinger transformations removing apparent singularities at the points d and b, c , and a respectively,

$$\begin{aligned} S_1(\lambda_1) &= \mu_1 J_{db} + 1/\mu_1 J_{bd}, & \mu_1 &= \sqrt{\frac{\lambda_1 - d}{\lambda_1 - b}}, \\ S_2(\lambda_1) &= \mu_2 J_{ac} + 1/\mu_2 J_{ca}, & \mu_2 &= \sqrt{\frac{\lambda_1 - a}{\lambda_1 - c}}, \\ S_3(\lambda_1) &= \mu_3 J_{\infty a} + 1/\mu_3 J_{a\infty}, & \mu_3 &= \sqrt{\lambda_1 - a}, \end{aligned}$$

$$J_{bd} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad J_{db} = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix},$$

$$J_{ac} = \frac{1}{20(a-b)} \begin{pmatrix} 13a+7d-20b & 7(a-d) \\ 13a+7d-20b & 7(a-d) \end{pmatrix}, \quad J_{ca} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - J_{ac},$$

$$J_{ac}[1, 2] = 1 - J_{ac}[1, 1] = \frac{609}{340} - \frac{4851}{3400}\zeta - \frac{1827}{3400}\zeta^2 - \frac{1701}{34000}\zeta^3,$$

$$J_{a\infty} = \frac{1}{300} \begin{pmatrix} 251 & 49 \\ 251 & 49 \end{pmatrix}, \quad J_{\infty a} = \frac{1}{300} \begin{pmatrix} 49 & -49 \\ -251 & 251 \end{pmatrix};$$

$$Q_\infty = \frac{1}{\sqrt{-\frac{251}{49} - Q}} \begin{pmatrix} 1 & 1 \\ Q & -\frac{251}{49} \end{pmatrix}, \quad Q = \frac{1}{7261} (329161 - 288000\zeta - 106920\zeta^2 - 9720\zeta^3),$$

for the value of ζ chosen at the end of Subsection 5.1 $J_{ac}[1, 2] = 7/60 + i7\sqrt{5}/150$, $Q = (-7439 + i5400\sqrt{15})/7261$; the number $p = \frac{8353}{2205} - \frac{32}{9}\zeta - \frac{33}{25}\zeta^2 - \frac{3}{25}\zeta^3$, in particular, for the choice of the root ζ considered at the end of Subsection 5.1, $p = -\frac{18}{49} + i\frac{1}{\sqrt{15}}$; the function $\Phi_1(\lambda_1; \frac{1}{3}, \frac{1}{3}, \frac{1}{2})$ is exactly the same as the one in the previous subsection, i.e., it solves Eq. (2.1) with $\lambda \rightarrow \lambda_1$ and the matrices A and B changed respectively by A_1 and B_1 which are given in Eq. (5.5); Eq. (2.2) with $(\lambda, \alpha, \beta, \delta)$ changed to $(\lambda_1, \alpha_1, \beta_1, \delta_1)$ given in Eq. (5.6) defines an explicit representation for $\Phi_1(\lambda_1; \frac{1}{3}, \frac{1}{3}, \frac{1}{2})$; finally, the matrix V , which is independent of λ_1 , can be calculated in the same way as in the previous Subsection,

$$V = C_0^{-1}(-1/12, -7/12, 2/3) D_0^{-1} q^{\frac{\sigma_3}{6}},$$

where the number q and the matrix $C_0^{-1}(-1/12, -7/12, 2/3)$ are defined in Eqs. (5.8) and (5.7) respectively, whilst the diagonal (!) matrix D_0 is different,

$$\begin{aligned} D_0 &= S_1(0)S_2(0)S_3(0)Q_\infty p^{\frac{\sigma_3}{2}} G_0(-1/12, -7/12, 2/3) = \begin{pmatrix} s & 0 \\ 0 & \frac{2}{3s} \end{pmatrix}, \\ s &= \sqrt{-\frac{46}{3} + \frac{40}{3}\zeta + \frac{99}{20}\zeta^2 + \frac{9}{20}\zeta^3}. \end{aligned}$$

For our choice of ζ at the end of Subsection 5.1 $s = 1/4(\sqrt{10} - i\sqrt{6})$.

RS -transformation found in this subsection connects hypergeometric functions corresponding to the second and fourteen cases of the Schwartz table.

6 $RS_3^2(5+1|2+2+2|3+2+1)$

6.1 $R(5+1|2+2+2|3+2+1)$

The rational transformation of the variable λ , which we denote as $R(5+1|2+2+2|3+2+1)$ reads,

$$\lambda = \frac{4}{27} \frac{\lambda_1(16\lambda_1 + 9)^5}{(\lambda_1 - 1)^2(128\lambda_1 - 3)^3}, \quad \lambda - 1 = \frac{(2048\lambda_1^3 - 10944\lambda_1^2 + 2619\lambda_1 + 27)^2}{27(\lambda_1 - 1)^2(128\lambda_1 - 3)^3}. \quad (6.1)$$

We denote the root and pole of the largest multiplicities, five and three respectively, of the R -transformation as

$$a = -\frac{9}{16} = -0.5625, \quad b = \frac{3}{128} = 0.0234375.$$

The double roots of $\lambda - 1$ are denoted as c_1 , c_2 , and c_3 . Their explicit values are

$$\begin{aligned} c_1 &= \frac{57}{32} + \frac{75}{32}\sqrt{2}\cos\left(\frac{1}{3}\arctan\left(\frac{1}{7}\right)\right), \\ c_2 &= \frac{57}{32} - \frac{75}{64}\sqrt{2}\cos\left(\frac{1}{3}\arctan\left(\frac{1}{7}\right)\right) - \frac{75}{64}\sqrt{6}\sin\left(\frac{1}{3}\arctan\left(\frac{1}{7}\right)\right), \\ c_3 &= \frac{57}{32} - \frac{75}{64}\sqrt{2}\cos\left(\frac{1}{3}\arctan\left(\frac{1}{7}\right)\right) + \frac{75}{64}\sqrt{6}\sin\left(\frac{1}{3}\arctan\left(\frac{1}{7}\right)\right). \end{aligned}$$

The first digits of these numbers in the floating-point representation are as follows,

$$c_1 = 5.0921060625\dots, \quad c_2 = -0.0098990450\dots, \quad c_3 = 0.2615429824\dots$$

According to Section 3 with this R -transformation one can associate four different classes of the RS -transformations corresponding to the following θ -triples:

$$\left(\frac{1}{5}, \frac{1}{2}, \frac{1}{3}\right), \quad \left(\frac{2}{5}, \frac{1}{2}, \frac{1}{3}\right), \quad \left(\frac{1}{5}, \frac{1}{2}, \frac{1}{2}\right), \quad \text{and} \quad \left(\frac{2}{5}, \frac{1}{2}, \frac{1}{2}\right).$$

These transformations are considered in the following subsections.

6.2 $RS_3^2\left(\begin{array}{c|c|c} 1/5 & -1/2 & 1/3 \\ 5+1 & 2+2+2 & 3+2+1 \end{array}\right)$

The choice of the parameters which corresponds to this transformation is exactly the same as in Subsection 5.2:

$$\alpha = \frac{19}{60}, \quad \beta = -\frac{1}{60}, \quad \delta = \frac{4}{5}.$$

Clearly that the corresponding matrices A and B in Eq. (2.1) also coincide with those given in Eq. (5.3). The RS -transformation reads,

$$\Phi\left(\lambda; \frac{1}{5}, -\frac{1}{2}, \frac{1}{3}\right) = S_1(\lambda_1)S_2(\lambda_1)S_3(\lambda_1)\Phi_1\left(\lambda_1; -\frac{1}{5}, -\frac{1}{3}, \frac{1}{3}\right)\left(\frac{27}{2}\right)^{\frac{\sigma_3}{3}}, \quad (6.2)$$

where λ is given by the rational transformation (6.1), λ_1 belongs to a neighborhood of ∞ which does not contain the points $0, 1, a, b, c_1, c_2$, and c_3 ; $S_1(\lambda_1)$, $S_2(\lambda_1)$, and $S_3(\lambda_1)$ are the matrices defining Schlesinger transformations removing apparent singularities at the points a and c_1 , b and c_2 , and c_3 respectively,

$$\begin{aligned} S_1(\lambda_1) &= \mu_1 J_{ac_1} + 1/\mu_1 J_{c_1 a}, & \mu_1 &= \sqrt{\frac{\lambda_1 - a}{\lambda_1 - c_1}}, \\ S_2(\lambda_1) &= \mu_2 J_{bc_2} + 1/\mu_2 J_{c_2 b}, & \mu_2 &= \sqrt{\frac{\lambda_1 - b}{\lambda_1 - c_2}}, \\ S_3(\lambda_1) &= \mu_3 J_{1c_3} + 1/\mu_3 J_{c_3 1}, & \mu_3 &= \sqrt{\frac{\lambda_1 - 1}{\lambda_1 - c_3}}, \end{aligned}$$

$$\begin{aligned} J_{c_1 a} &= \frac{1}{20} \begin{pmatrix} 1 & -1 \\ -19 & 19 \end{pmatrix}, & J_{ac_1} &= \frac{1}{20} \begin{pmatrix} 19 & 1 \\ 19 & 1 \end{pmatrix}, \\ J_{bc_2} &= \frac{1}{5} \begin{pmatrix} 4 - \mathcal{C} & 1 + \mathcal{C} \\ 4 - \mathcal{C} & 1 + \mathcal{C} \end{pmatrix}, & J_{c_2 b} &= \frac{1}{5} \begin{pmatrix} 1 + \mathcal{C} & -1 - \mathcal{C} \\ -4 + \mathcal{C} & 4 - \mathcal{C} \end{pmatrix}, \\ \mathcal{C} &= \sqrt{2} \cos \left(\frac{1}{3} \arctan \left(\frac{1}{7} \right) \right), & \mathcal{S} &= \sqrt{2} \sin \left(\frac{1}{3} \arctan \left(\frac{1}{7} \right) \right), \\ J_{1c_3} &= \begin{pmatrix} x & xy \\ \frac{1-x}{y} & 1-x \end{pmatrix}, & J_{c_3 1} &= \begin{pmatrix} 1-x & -xy \\ -\frac{1-x}{y} & x \end{pmatrix}, \\ x &= \frac{7}{10} + \frac{3\sqrt{3}}{10} \mathcal{C} \mathcal{S} + \frac{3\sqrt{6}}{20} \mathcal{S} - \frac{3\sqrt{2}}{20} \mathcal{C} + \frac{3}{10} \mathcal{C}^2, \\ y &= \frac{3}{\sqrt{2}} \frac{(\sqrt{2}\mathcal{C} + 1)(12\mathcal{C}^3 + 18\sqrt{2}\mathcal{C}^2 - 38\mathcal{C} + 5\sqrt{2} - 12\sqrt{3}\mathcal{S}\mathcal{C}^2 + 6\sqrt{6}\mathcal{S}\mathcal{C} - 8\sqrt{3}\mathcal{S})}{(66\sqrt{3}\mathcal{S}\mathcal{C} + 42\sqrt{6}\mathcal{S} + 125 + 12\sqrt{2}\mathcal{C} + 120\mathcal{C}^2 - 72\sqrt{2}\mathcal{C}^3 - 36\mathcal{C}^4 + 36\sqrt{3}\mathcal{S}\mathcal{C}^3)}, \end{aligned}$$

The function $\Phi_1(\lambda_1; -\frac{1}{5}, -\frac{1}{3}, \frac{1}{3})$ solves Eq. (2.1) with $\lambda \rightarrow \lambda_1$ and the matrices A and B changed respectively by A_1 and B_1 ,

$$A_1 = \frac{1}{300} \begin{pmatrix} -9 & 9 \\ 91 & 9 \end{pmatrix}, \quad B_1 = \frac{1}{300} \begin{pmatrix} -41 & -9 \\ -91 & 41 \end{pmatrix}, \quad (6.3)$$

and is given by Eq. (2.2) with $(\lambda, \alpha, \beta, \delta)$ changed to $(\lambda_1, \alpha_1, \beta_1, \delta_1)$, where

$$\alpha_1 = \frac{13}{30}, \quad \beta_1 = \frac{1}{10}, \quad \delta_1 = \frac{6}{5}. \quad (6.4)$$

Some transformations of the sixth order are compositions of the quadratic and cubic transformations. Let us address this question in more detail to check that this does not apply to the transformation constructed in this subsection. Let us list corresponding θ -triples which can be obtained by application of the quadratic and cubic transformations to the θ -triple $(1/2, 1/3, 1/5)$, which is equivalent to the one considered in this subsection. We don't indicate below explicitly transformations (3.1)–(3.5) which, of course, essential at the level of the equations (functions):

1. A quadratic transform (4.2),

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5} \right) \rightarrow \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{3} \right);$$

2. A combination of the quadratic transform (4.2) with cubic (2),

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right) \rightarrow \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \rightarrow \left(\frac{3}{5}, \frac{3}{5}, \frac{3}{5}\right);$$

3. Combinations of the cubic transform (3a) with quadratic ones (4.2),

(a)

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right) \rightarrow \left(\frac{1}{2}, \frac{1}{5}, \frac{2}{5}\right) \rightarrow \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right);$$

(b)

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right) \rightarrow \left(\frac{1}{2}, \frac{1}{5}, \frac{2}{5}\right) \rightarrow \left(\frac{2}{5}, \frac{2}{5}, \frac{3}{5}\right).$$

These transformations connect hypergeometric functions corresponding to the following cases of the Schwartz table: 6 and 8; 6, 7, and 11; 6, 9, and 13; and 6, 9, and 11, respectively. RS -transformation found in this subsection connects hypergeometric functions corresponding to the cases 6 and 12 of the Schwartz table, therefore it can't be obtained as a combination of the lower order transformations.

Remark 6.1 *For exact correspondence one should notice that our notation θ_0 , θ_1 , and θ_∞ are related with the variables λ , μ , and ν introduced by H. A. Schwartz (see [2]) as follows $\lambda = \theta_0$, $\nu = \theta_1$, $\mu = \theta_\infty - 1$.*

$$\mathbf{6.3} \quad RS_3^2 \left(\begin{array}{c|c|c} 2/5 & -1/2 & 1/3 \\ \hline 5+1 & 2+2+2 & 3+2+1 \end{array} \right)$$

The choice of the parameters which correspond to this transformation is as follows,

$$\alpha = \frac{13}{60}, \quad \beta = -\frac{7}{60}, \quad \delta = \frac{3}{5}.$$

The corresponding matrices A and B in Eq. (2.1) read,

$$A = \frac{1}{1200} \begin{pmatrix} -19 & 119 \\ 481 & 19 \end{pmatrix}, \quad B = \frac{1}{1200} \begin{pmatrix} -181 & -119 \\ -481 & 181 \end{pmatrix}. \quad (6.5)$$

The form of the RS -transformation is as follows,

$$\Phi \left(\lambda; \frac{2}{5}, -\frac{1}{2}, \frac{1}{3} \right) = S_1(\lambda_1) S_2(\lambda_1) S_3(\lambda_1) \left(\frac{\sqrt{11}}{6} \right)^{\sigma_3} \Phi_1 \left(\lambda_1; \frac{2}{5}, \frac{2}{3}, \frac{1}{3} \right) \left(\frac{9\sqrt[3]{4}}{\sqrt{11}} \right)^{\sigma_3}, \quad (6.6)$$

where λ is given by the rational transformation (6.1), λ_1 belongs to a neighborhood of ∞ which does not contain the points 0, 1, a , b , c_1 , c_2 , and c_3 ; $S_1(\lambda_1)$, $S_2(\lambda_1)$, and $S_3(\lambda_1)$ are the matrices defining Schlesinger transformations removing apparent singularities at the points b and c_1 , c_2 , and c_3 respectively,

$$\begin{aligned} S_1(\lambda_1) &= \mu_1 J_{bc_1} + 1/\mu_1 J_{c_1b}, & \mu_1 &= \sqrt{\frac{\lambda_1 - b}{\lambda_1 - c_1}}, \\ S_2(\lambda_1) &= \mu_2 J_{ac_2} + 1/\mu_2 J_{c_2a}, & \mu_2 &= \sqrt{\frac{\lambda_1 - a}{\lambda_1 - c_2}}, \\ S_3(\lambda_1) &= \mu_3 J_{ac_3} + 1/\mu_3 J_{c_3a}, & \mu_3 &= \sqrt{\frac{\lambda_1 - a}{\lambda_1 - c_3}}, \end{aligned}$$

$$\begin{aligned}
J_{c_1b} &= \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}, & J_{bc_1} &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \\
J_{ac_2} &= \frac{1}{5} \begin{pmatrix} -2-7\mathcal{C} & 7+7\mathcal{C} \\ -2-7\mathcal{C} & 7+7\mathcal{C} \end{pmatrix}, & J_{c_2a} &= \frac{1}{5} \begin{pmatrix} 7+7\mathcal{C} & -7-7\mathcal{C} \\ 2+7\mathcal{C} & -2-7\mathcal{C} \end{pmatrix}, \\
\mathcal{C} &= \sqrt{2} \cos \left(\frac{1}{3} \arctan \left(\frac{1}{7} \right) \right), & \mathcal{S} &= \sqrt{2} \sin \left(\frac{1}{3} \arctan \left(\frac{1}{7} \right) \right), \\
J_{ac_3} &= \begin{pmatrix} 1-x & x \\ 1-x & x \end{pmatrix}, & J_{c_3a} &= \begin{pmatrix} x & -x \\ x-1 & 1-x \end{pmatrix},
\end{aligned}$$

$$x = \frac{7(3+\sqrt{3})}{600}((20\sqrt{3}-24)\mathcal{C}^2 - 12\mathcal{C}\mathcal{S} - 10\sqrt{3} + 12 + (7\sqrt{6} - 9\sqrt{2})\mathcal{C} + (\sqrt{6} + 3\sqrt{2})\mathcal{S}),$$

The function $\Phi_1(\lambda_1; \frac{2}{5}, \frac{2}{3}, \frac{1}{3})$ solves Eq. (2.1) with $\lambda \rightarrow \lambda_1$ and the matrices A and B changed respectively by A_1 and B_1 ,

$$A_1 = \frac{1}{300} \begin{pmatrix} 39 & -189 \\ -11 & -39 \end{pmatrix}, \quad B_1 = \frac{1}{300} \begin{pmatrix} -89 & 189 \\ 11 & 89 \end{pmatrix},$$

and is given by Eq. (2.2) with $(\lambda, \alpha, \beta, \delta)$ changed to $(\lambda_1, \alpha_1, \beta_1, \delta_1)$, where

$$\alpha_1 = -\frac{11}{30}, \quad \beta_1 = -\frac{7}{10}, \quad \delta_1 = \frac{3}{5}.$$

As in the previous subsection we present here the list of the θ -triples which can be obtained by application of the quadratic and cubic transformations to the equations (functions) corresponding to the θ -triple $(1/2, 1/3, 2/5)$ to prove that our transformation is not a composition of the lower order ones. As before we don't indicate explicitly in the list below transformations (3.1)–(3.5) and keep in mind Remark 6.1:

1. A quadratic transform (4.2),

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{5} \right) \rightarrow \left(\frac{2}{5}, \frac{2}{5}, \frac{1}{3} \right);$$

2. A combination of the quadratic transform (4.2) with cubic (2),

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{5} \right) \rightarrow \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{5} \right) \rightarrow \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5} \right);$$

3. Combinations of the cubic transform (3a) with quadratic ones (4.2),

(a)

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{5} \right) \rightarrow \left(\frac{1}{2}, \frac{2}{5}, \frac{1}{5} \right) \rightarrow \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5} \right);$$

(b)

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{5} \right) \rightarrow \left(\frac{1}{2}, \frac{2}{5}, \frac{1}{5} \right) \rightarrow \left(\frac{2}{5}, \frac{2}{5}, \frac{3}{5} \right).$$

These transformations connect hypergeometric functions corresponding to the following cases of the Schwartz table: 14 and 15; 14, 12, and 13; 14, 9, and 13; and 14, 9, and 11, respectively. *RS*-transformation found in this subsection connects hypergeometric functions corresponding to the cases 14 and 7 of the Schwartz table, therefore it can't be obtained as a combination of the lower order transformations.

$$6.4 \quad RS_3^2 \left(\begin{array}{c|c|c} 1/5 & 1/2 & 1/2 \\ \hline 5+1 & 2+2+2 & 3+2+1 \end{array} \right)$$

In this case it is convenient to rescale R -transformation (6.1),

$$\lambda = \frac{1}{64} \frac{\lambda_1(\lambda_1 + 24)^5}{(\lambda_1 - 1)^3(3\lambda_1 - 128)^2}, \quad \lambda - 1 = \frac{(\lambda_1^3 - 228\lambda_1^2 + 2328\lambda_1 + 1024)^2}{64(\lambda_1 - 1)^3(3\lambda_1 - 128)^2}. \quad (6.7)$$

We denote the root of the R -transformation of the fifth order, and the pole of the second as

$$a = -24, \quad b = \frac{128}{3} = 42.(6), \quad (6.8)$$

respectively. The double roots of $\lambda - 1$ are denoted as c_1 , c_2 , and c_3 . Their explicit values are

$$c_1 = 76 + 100\sqrt{2} \cos \left(\frac{1}{3} \arctan \left(\frac{1}{7} \right) \right), \quad (6.9)$$

$$c_2 = 76 - 50\sqrt{2} \cos \left(\frac{1}{3} \arctan \left(\frac{1}{7} \right) \right) - 50\sqrt{6} \sin \left(\frac{1}{3} \arctan \left(\frac{1}{7} \right) \right), \quad (6.10)$$

$$c_3 = 76 - 50\sqrt{2} \cos \left(\frac{1}{3} \arctan \left(\frac{1}{7} \right) \right) + 50\sqrt{6} \sin \left(\frac{1}{3} \arctan \left(\frac{1}{7} \right) \right). \quad (6.11)$$

The first digits of these numbers in the floating-point representation are as follows,

$$c_1 = 217.2631920020 \dots, \quad c_2 = -0.4223592539 \dots, \quad c_3 = 11.1591672518 \dots$$

The choice of the parameters corresponding to this RS -transformation is as follows,

$$\alpha = -\frac{1}{10}, \quad \beta = -\frac{3}{5}, \quad \delta = \frac{4}{5}. \quad (6.12)$$

The corresponding matrices A and B in Eq. (2.1) read,

$$A = \frac{1}{50} \begin{pmatrix} -1 & -24 \\ -1 & 1 \end{pmatrix}, \quad B = \frac{1}{100} \begin{pmatrix} -23 & 24 \\ 2 & 23 \end{pmatrix}. \quad (6.13)$$

The RS -transformation can be written in the following form,

$$\Phi \left(\lambda; \frac{1}{5}, \frac{1}{2}, \frac{1}{2} \right) = S_1(\lambda_1) S_2(\lambda_1) S_3(\lambda_1) \left(i2\sqrt{6} \right)^{\sigma_3} \Phi_1 \left(\lambda_1; \frac{1}{5}, \frac{1}{2}, \frac{1}{2} \right) \left(-i2\sqrt{6} \right)^{\sigma_3}, \quad (6.14)$$

where λ is given by the rational transformation (6.7), λ_1 belongs to a neighborhood of ∞ which does not contain the points 0 , 1 , a , b , c_1 , c_2 , and c_3 ; $S_1(\lambda_1)$, $S_2(\lambda_1)$, and $S_3(\lambda_1)$ are the matrices defining Schlesinger transformations removing apparent singularities at the points a and c_1 , b and c_2 , and c_3 respectively,

$$\begin{aligned} S_1(\lambda_1) &= \mu_1 J_{ac_1} + 1/\mu_1 J_{c_1a}, & \mu_1 &= \sqrt{\frac{\lambda_1 - a}{\lambda_1 - c_1}}, \\ S_2(\lambda_1) &= \mu_2 J_{bc_2} + 1/\mu_2 J_{c_2b}, & \mu_2 &= \sqrt{\frac{\lambda_1 - b}{\lambda_1 - c_2}}, \\ S_3(\lambda_1) &= \mu_3 J_{1c_3} + 1/\mu_3 J_{c_31}, & \mu_3 &= \sqrt{\frac{\lambda_1 - 1}{\lambda_1 - c_3}}, \end{aligned}$$

$$\begin{aligned}
J_{c_1 a} &= \frac{1}{30} \begin{pmatrix} 6 & 144 \\ 1 & 24 \end{pmatrix}, & J_{ac_1} &= \frac{1}{30} \begin{pmatrix} 24 & -144 \\ -1 & 6 \end{pmatrix}, \\
J_{bc_2} &= \begin{pmatrix} \frac{7}{10} - \frac{3}{10}\mathcal{C} & -\frac{36}{5} - \frac{36}{5}\mathcal{C} \\ -\frac{7}{240} + \frac{1}{80}\mathcal{C} & \frac{3}{10} + \frac{3}{10}\mathcal{C} \end{pmatrix}, & J_{c_2 b} &= \begin{pmatrix} \frac{3}{10} + \frac{3}{10}\mathcal{C} & \frac{36}{5} + \frac{36}{5}\mathcal{C} \\ \frac{7}{240} - \frac{1}{80}\mathcal{C} & \frac{7}{10} - \frac{3}{10}\mathcal{C} \end{pmatrix}, \\
\mathcal{C} &= \sqrt{2} \cos \left(\frac{1}{3} \arctan \left(\frac{1}{7} \right) \right), & \mathcal{S} &= \sqrt{2} \sin \left(\frac{1}{3} \arctan \left(\frac{1}{7} \right) \right), \\
J_{1c_3} &= \begin{pmatrix} x & xy \\ (1-x)/y & 1-x \end{pmatrix}, & J_{c_3 1} &= \begin{pmatrix} 1-x & -xy \\ (x-1)/y & x \end{pmatrix}, \\
x &= -\frac{1}{5} - \frac{3}{5}\mathcal{C} + \frac{3}{5}\sqrt{3}\mathcal{S} + \frac{3}{5}\sqrt{3}\mathcal{C}\mathcal{S} + \frac{3}{5}\mathcal{C}^2, \\
y &= 72 \frac{(\mathcal{C}+1)(6\mathcal{C}^3 - 6\sqrt{3}\mathcal{S}\mathcal{C}^2 + 18\mathcal{C}^2 + 6\sqrt{3}\mathcal{C}\mathcal{S} - 23\mathcal{C} + 7\sqrt{3}\mathcal{S} - 20)}{18\mathcal{C}^4 - 18\sqrt{3}\mathcal{C}^3\mathcal{S} + 72\mathcal{C}^3 - 30\mathcal{C}^2 + 24\sqrt{3}\mathcal{C}\mathcal{S} + 6\sqrt{3}\mathcal{S} - 114\mathcal{C} - 55}.
\end{aligned}$$

The function $\Phi_1(\lambda_1; \frac{1}{5}, \frac{1}{2}, \frac{1}{2})$ solves Eq. (2.1) with $\lambda \rightarrow \lambda_1$ and the same matrices A and B as for the function $\Phi(\lambda; \frac{1}{5}, \frac{1}{2}, \frac{1}{2})$ (see Eq. (6.13)) and is given by Eq. (2.2) with λ changed to λ_1 , whilst the parameters α , β , and δ are the same as before the transformation and given by Eq. (6.12). This transformation is related with the hypergeometric function, $F(-1/10, 1/10; 1/2; 1/\lambda)$. Its trigonometric form is well known [1, 2], $F(-1/10, 1/10; 1/2; \sin^2 t) = \cos(\frac{1}{5}t)$. The algebraic form of this function is as follows,

$$F\left(-\frac{1}{10}, \frac{1}{10}; \frac{1}{2}; \frac{1}{\lambda}\right) = \frac{1}{2} \left(\sqrt{-\frac{1}{\lambda}} + \sqrt{1 - \frac{1}{\lambda}} \right)^{\frac{1}{5}} + \frac{\left((2 - \frac{4}{\lambda}) \left(\sqrt{-\frac{1}{\lambda}} + \sqrt{1 - \frac{1}{\lambda}} \right)^2 - 1 \right)^{\frac{1}{5}}}{2 \left(\sqrt{-\frac{1}{\lambda}} + \sqrt{1 - \frac{1}{\lambda}} \right)}.$$

The formula is valid in a proper neighborhood of $\lambda = \infty$.

It is worth to notice, that there are higher-order transformations related with this function which can be obtained as iterations of quadratic and cubic transformations (as in the previous subsections we don't indicate explicitly transformations (3.1)–(3.5)):

1. A fourth order transform, as an iteration of two quadratic transformations (4.2);
2. A sixth order transform, as the successive (in arbitrary order) quadratic (4.2) and cubic (3b) transforms;
3. A ninth order transform, as an iteration of the cubic (3b) transforms.

All these three cases, in terms of the θ -triples read identically,

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{5} \right) \rightarrow \left(\frac{1}{2}, \frac{1}{2}, \frac{3}{5} \right) \rightarrow \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{5} \right),$$

although in terms of the hypergeometric function they are very different. A comparison of the R -transforms, which correspond to the sixth order transformations obtained in this subsection, and the one described in item 2 shows that it is two different transformations of the six order. All these auto-transformations corresponds to the first case of the Schwartz table.

A situation with RS -transformations corresponding to the triple $(2/5, 1/2, 1/2)$ is absolutely parallel with the one considered in this subsection: corresponding hypergeometric function, $F(-1/5, 1/5; 1/2; \sin^2 t) = \cos(\frac{2}{5}t)$, is also a superposition of elementary functions, there are auto-transformations of the fourth and sixth order which are superposition of the quadratic and cubic transformations. Moreover the hypergeometric functions corresponding to the triples $(2/5, 1/2, 1/2)$ and $(1/5, 1/2, 1/2)$ are related by a quadratic transformation. We omit here explicit formulas for the corresponding auto- RS -transformation related with the R -transform (6.1), although it is new it does not seem to add much knowledge to the subject.

7 Concluding remarks

As it was mentioned above most of the higher order transformations relate hypergeometric functions corresponding to the parameters from the Schwartz table. At the same time it is clear that there are higher order transformations for the hypergeometric functions whose parameters are different from the ones included in the Schwartz table. The quadratic and cubic transformations (see Section 4), which contain an arbitrary parameter, are the most known of them. There are also transformations of the fourth, fifth, and sixth order with a parameter which also run out of the Schwartz table. For example, combinations of the cubic and quadratic transformations considered in Section 4 give sixth order transformations which relate hypergeometric functions corresponding to the following θ parameters:

$$\begin{aligned} \left(\frac{1}{2}, \frac{1}{3}, \theta_\infty\right) &\longrightarrow (\theta_\infty, \theta_\infty, 4\theta_\infty - 1), \\ \left(\frac{1}{2}, \frac{1}{3}, \theta_\infty\right) &\longrightarrow (2\theta_\infty, 2\theta_\infty, 2\theta_\infty - 1). \end{aligned}$$

Here we would like to present interesting RS -transformations of the rank 8 which are based on the following R -transformation,

$$\lambda = \frac{\rho\lambda_1(\lambda_1 - a)^7}{(\lambda_1 - 1)(\lambda_1 - c_1)^3(\lambda_1 - c_2)^3}, \quad \lambda - 1 = \frac{\rho(\lambda_1 - b_1)^2(\lambda_1 - b_2)^2(\lambda_1 - b_3)^2(\lambda_1 - b_4)^2}{(\lambda_1 - 1)(\lambda_1 - c_1)^3(\lambda_1 - c_2)^3}. \quad (7.1)$$

We were not able to find the numbers ρ , a , c_1 , c_2 , b_1 , b_2 , b_3 , and b_4 explicitly. However, an application of the Groebner package on MAPLE 6 code shows that transformation (7.1) really exists. Moreover, we were able to calculate this transformation numerically. The following list presents first 20 correct digits after the decimal point in the floating-point representation of the numbers defining transformation (7.1):

$$\begin{aligned} \rho &= 1/112 - i0.04639421805988064179 \dots, \\ a &= 0.27551020408163265306 \dots - i0.68928552546108382089 \dots, \\ c_1 &= 0.00025904071151396454 \dots + i0.02275992796424688660 \dots, \\ c_2 &= 0.65407769398236358647 \dots + i0.93826316041899497905 \dots, \\ b_1 &= 0.00004664085318617660 \dots - i0.00965813289425897678 \dots, \\ b_2 &= 0.03098261455781643319 \dots + i0.24699252359291090010 \dots, \\ b_3 &= 1.01560634234261281988 \dots + i0.12947315141387009072 \dots, \end{aligned}$$

$$b_4 = 1.91765011653209885603 \dots + i7.61299796418694837399 \dots$$

As far as R -transformation is known it is straightforward to write an associated RS -transformations. In this case, we omit here this formulas, since we can write them only in terms of unknown numbers ρ , a , c_1 , c_2 , b_1 , b_2 , b_3 , which can be defined as a solution of a system of polynomial equations. In such form the formulas for the S -transformations look quite cumbersome. We can actually associate three different classes of RS -transformations with the R -transform (7.1). These classes are defined by the RS -transformations whose action on the associated θ -triples read:

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{7}\right) \rightarrow \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{7}\right), \quad \left(\frac{1}{2}, \frac{1}{3}, \frac{2}{7}\right) \rightarrow \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{7}\right), \quad \left(\frac{1}{2}, \frac{1}{3}, \frac{3}{7}\right) \rightarrow \left(\frac{1}{3}, \frac{2}{3}, \frac{3}{7}\right).$$

Acting on these θ -triples by other RS -transformations of the rank ≤ 6 we arrive to a cluster (actually three clusters) of the θ -triples which do not intersect with the Schwartz table and whose corresponding hypergeometric functions are related via higher order transformations.

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